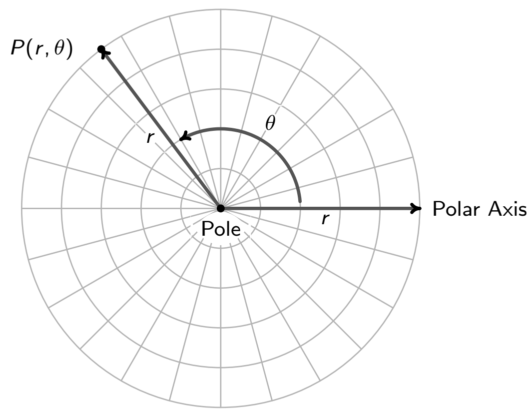
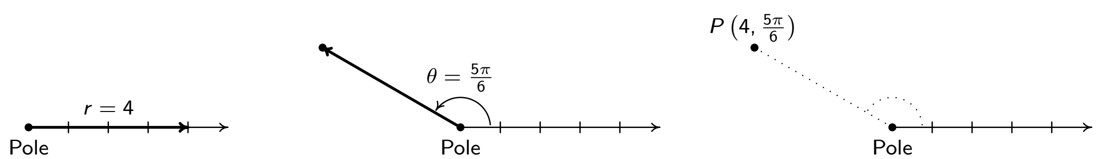
SECTION 12.2: POLAR EQUATIONS

In this section, we introduce a new system for assigning coordinates to points in the plane – as diagrammed above on the right. We start with an origin point, called the **pole**, and a ray called the **polar axis**. We locate a point P using two coordinates, (r, θ), where r represents a *directed* distance from the pole and θ is a measure of counter-clockwise rotation from the polar axis. Roughly speaking, the polar coordinates (r, θ) of a point measure 'how far out' the point is from the pole (that's r), and 'how far to rotate' from the polar axis, (that's θ).

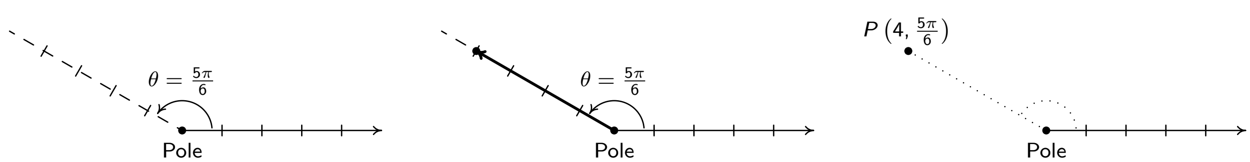


Polar Coordinates

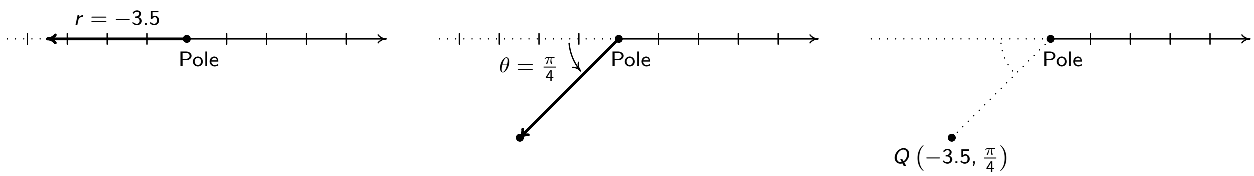
For example, if we wished to plot the point P with polar coordinates $\left(4, \frac{5\pi}{6}\right)$, we'd start at the pole, move out along the polar axis 4 units, then rotate $\frac{5\pi}{6}$ radians counter-clockwise.



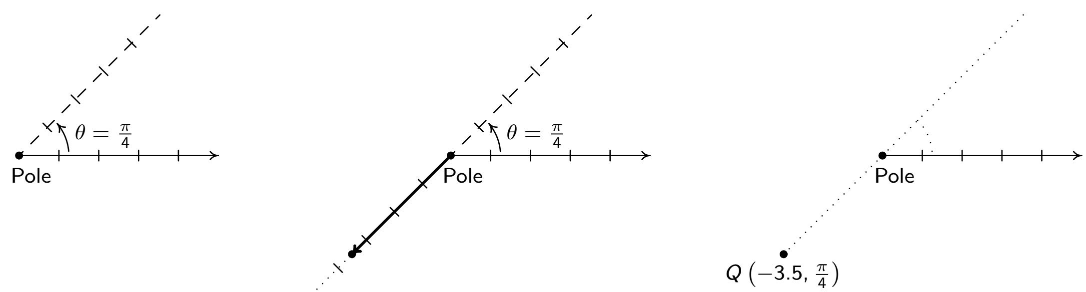
We may also visualize this process by thinking of the rotation first. To plot $P\left(4, \frac{5\pi}{6}\right)$ this way, we rotate $\frac{5\pi}{6}$ counter-clockwise from the polar axis, then move outwards from the pole 4 units. Essentially we are locating a point on the terminal side of $\frac{5\pi}{6}$ which is 4 units away from the pole.



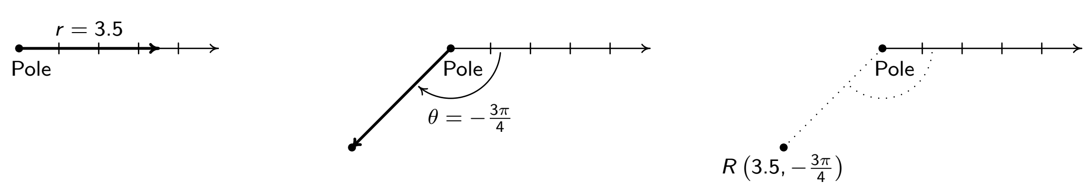
If $r < 0$, we begin by moving in the opposite direction on the polar axis from the pole. For example, to plot the point with polar coordinates $Q\left(-3.5, \frac{\pi}{4}\right)$ we have



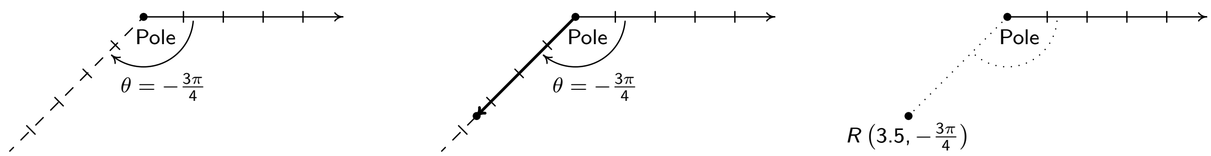
If we interpret the angle first, we rotate $\frac{\pi}{4}$ radians, then move back through the pole 3.5 units. Here we are locating a point 3.5 units away from the pole on the terminal side of $\frac{5\pi}{4}$, not $\frac{\pi}{4}$.



As you may have guessed, $\theta < 0$ means the rotation away from the polar axis is clockwise instead of counter-clockwise. Hence, to plot $R\left(3.5, -\frac{3\pi}{4}\right)$ we have the following.



From an 'angles first' approach, we rotate $-\frac{3\pi}{4}$ then move out 3.5 units from the pole. We see that R is the point on the terminal side of $\theta = -\frac{3\pi}{4}$ which is 3.5 units from the pole.



The points Q and R above are, in fact, the same point despite the fact that their polar coordinate representations are different. Unlike Cartesian coordinates where (a, b) and (c, d) represent the same point if and only if $a = c$ and $b = d$, a point can be represented by infinitely many polar coordinate pairs:

EQUIVALENT REPRESENTATIONS OF POINTS IN POLAR COORDINATES

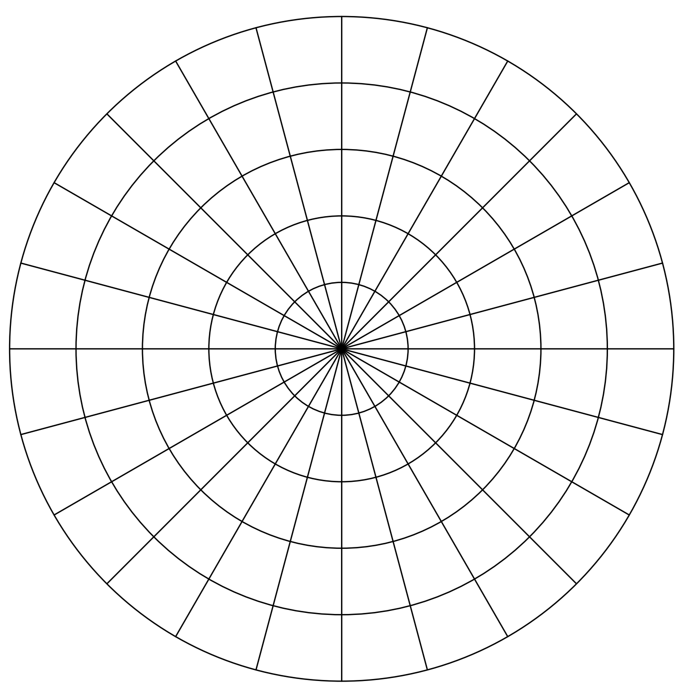
Suppose (r, θ) and (r', θ') are polar coordinates where $r \neq 0$, $r' \neq 0$ and the angles are measured in radians. Then (r, θ) and (r', θ') determine the same point P if and only if one of the following is true:

- $r' = r$ and $\theta' = \theta + 2\pi k$ for some integer k
- $r' = -r$ and $\theta' = \theta + (2k+1)\pi$ for some integer k

NOTE: All polar coordinates of the form $(0, \theta)$ represent the pole regardless of the value of θ.

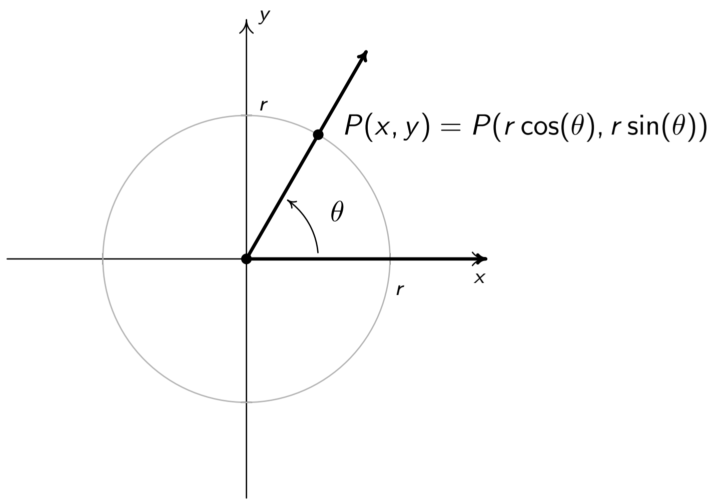
EXAMPLE 1: For each point in polar coordinates given below plot the point and then give two additional expressions for the point, one of which has $r > 0$ and the other with $r < 0$.

1. $P\left(2, \frac{7\pi}{6}\right)$
2. $P\left(5, -\frac{5\pi}{4}\right)$
3. $P\left(-4, \frac{2\pi}{3}\right)$
4. $P\left(-3, -3\pi\right)$



CONVERTING BETWEEN RECTANGULAR AND POLAR COORDINATES:

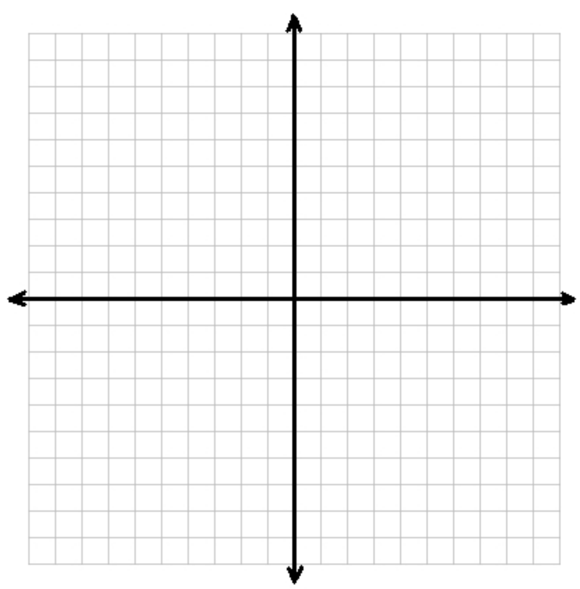
Suppose P is represented in rectangular coordinates as (x, y) and in polar coordinates as (r, θ). Then



- $x = r\cos(\theta)$ and $y = r\sin(\theta)$
- $x^2 + y^2 = r^2$ and $\tan(\theta) = \dfrac{y}{x}$ (provided $x \neq 0$)

EXAMPLE 2: Convert each point in rectangular coordinates given below into polar coordinates with $r \geq 0$ and $0 \leq \theta < 2\pi$. Check your answer by converting them back to rectangular coordinates.

1. $P\left(2, -2\sqrt{3}\right)$
2. $Q(-3, -3)$
3. $R(0, -3)$
4. $S(-3, 4)$



EXAMPLE 3: Convert each equation in rectangular coordinates into an equation in polar coordinates.

1. $x^2 + y^2 = 9$
2. $y = -x$
3. $x^2 + y^2 - 6y = 0$

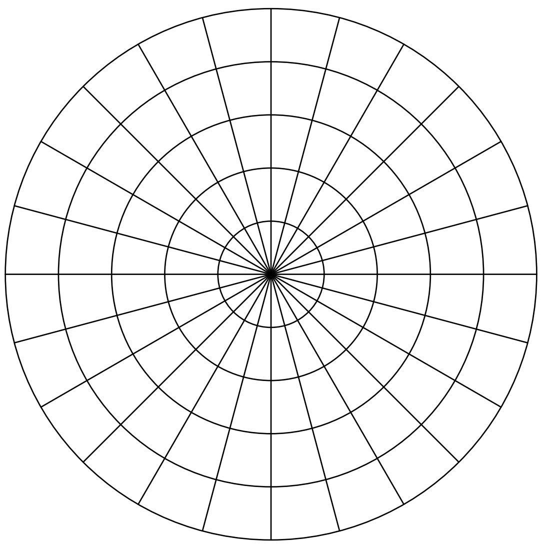
Ans:

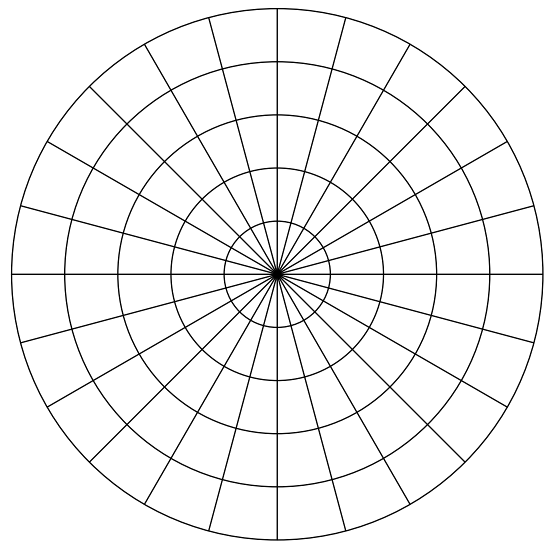
1. $r = 3$
2. $\theta = \dfrac{3\pi}{4}$
3. $r = 6\sin(\theta)$

GRAPHING IN POLAR COORDINATES:

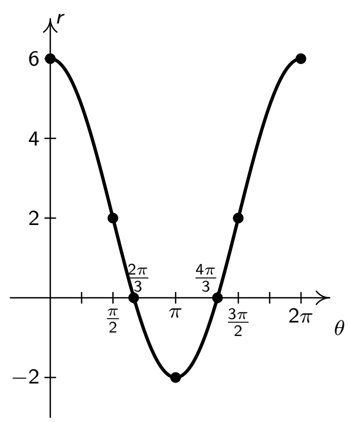
EXAMPLE 4: Graph the following polar equations in the xy-plane.

1. $r = 4$
2. $r = -3$
3. $\theta = \frac{5\pi}{4}$
4. $\theta = -\frac{3\pi}{2}$

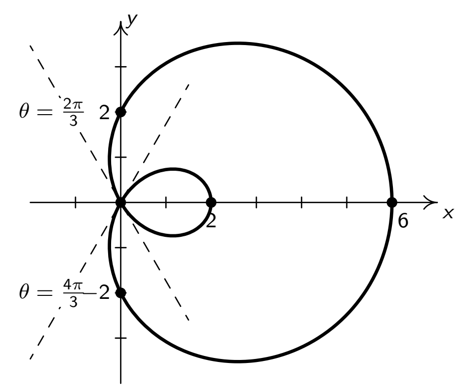




To graph more involved polar curves, it often helps to graph the relationship between r and θ in the θr-plane:



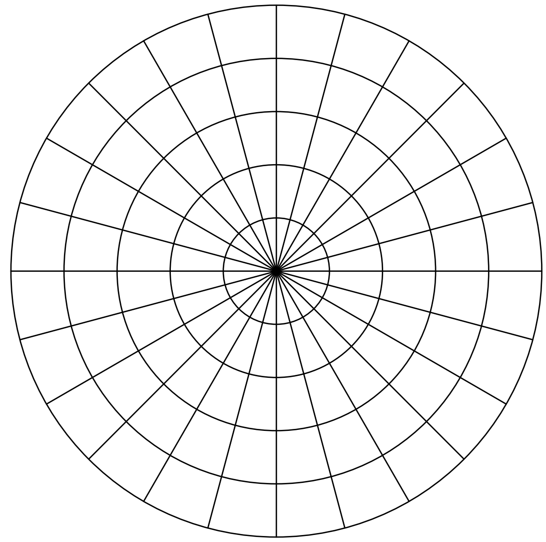
$r = 2 + 4\cos(\theta)$ in the θr-plane



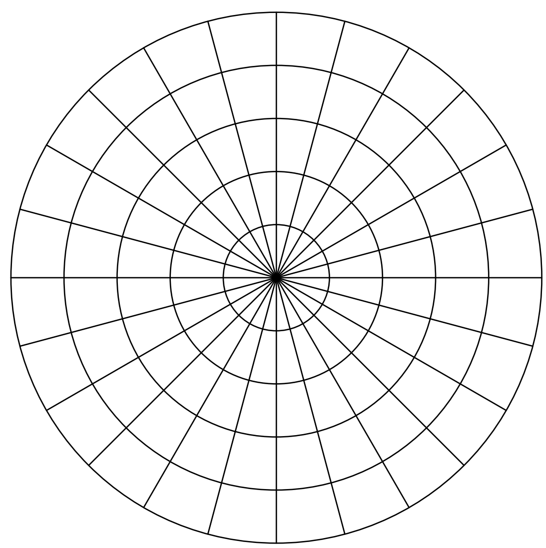
$r = 2 + 4\cos(\theta)$ in the xy-plane

EXAMPLE 5: Graph the following polar equations in the xy-plane.

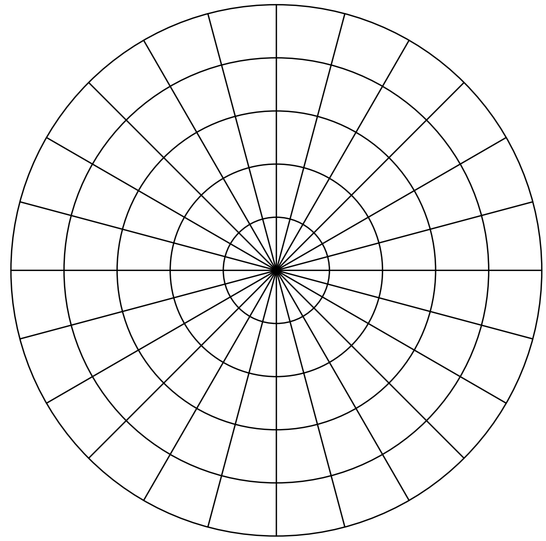
1. $r = 2 - \sin(\theta)$



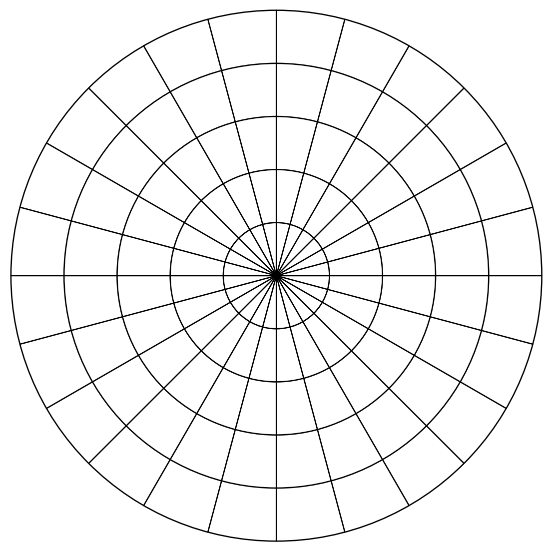
2. $r = 1 + 2\cos(\theta)$



3. $r = 4\sin(2\theta)$



4. $r^2 = 16\cos(2\theta)$



HOMEWORK: Section 12.2: 9 - 89 every other odd; 91*, 101*, 105*